Conformal Invariance of Brownian Motion

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1 Introduction

Brownian motion, a central topic in stochastic processes and stochastic calculus, models continuous random events. Originally discovered by Robert Brown in 1827 while observing the trajectories of pollen grains suspended in water, this type of motion is ubiquitous in physics and the world around us. Consequently, an understanding of its behavior is applicable to many practical applications. Surprisingly, complex analysis provides a natural framework to describe a crucial invariance property of Brownian motion: Lévy's Theorem. Furthermore, complex Brownian motion offers an alternative approach to proving theorems in complex analysis such as Louisville's theorem.

This paper assumes a general background in complex analysis and probability theory but does not require prior experience with stochastic processes. It will provide a basic overview of the relevant theory. Much of the information presented here is derived from the book "Conformally Invariant Processes in the Plane" by Lawler, which serves as an excellent resource for more in-depth explanations.

2 Brownian Motion

2.1 Definitions

We first begin by defining the basics.

Definition 2.1. A filtration on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is a family of ordered σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s < t$.

One can view $\mathcal{F}_s$ as a way of encoding historical information up to some time s. Consequently, by time t you have at least as much information about events as you had at time s.

We say a process H_t is adapted with respect to a filtration if H_t is $\mathcal{F}_t$ measurable. We say H_t is continuous if the map $t \mapsto H_t$ is almost surely continuous.

Definition 2.2. A stochastic process X_t is called a martingale with respect to a filtration $\mathcal{F}_t$, if for all t

- X_t is $\mathcal{F}_t$-measurable
- $\mathbb{E}[|X_t|] < \infty$
- For all $s \geq t$, $\mathbb{E}[X_s | \mathcal{F}_t] = X_t$

The last condition is the defining property of martingales, indicating that the conditional expectation of the process at a future time s, given the information up to time t, is equal to its current value at time t.

Definition 2.3. A standard 1 dimensional Brownian motion with respect to a filtration $\{\mathcal{F}_t\}$ is a family of real valued random variables B_t with the following properties:

- for $0 < s < t$ we have the random variable $B_t - B_s$ is $\mathcal{F}_t$ measurable, independent of B_s, and $B_t - B_s \sim N(0, t-s)$
- $t \mapsto B_t$ is almost surely continuous
- for standard Brownian Motion $B_0(\omega) = 0$. Other starting points are still Brownian motion, they are just not deemed standard.

Here B_t is a path that starts at 0, and as time progresses the certainty of the position diffuses. Furthermore the path is "memoryless" insofar as the prior path until a time t is independent of it's future path. $B_t(\omega)$ refers to a specific path at time t.

Definition 2.4. A complex Brownian motion with respect to a filtration $\mathcal{F}_t$ is defined as $B_t = B_t^1 + iB_t^2$ where B_t^1 and B_t^2 are independent 1 dimensional Brownian motions with respect to the same filtration.

2.2 Properties

2.2.1 Translation and Dilation Invariance

One key property of Brownian motion is translation and dilation invariance. Translation invariance is immediate. Changing the starting point does not affect the increments or continuity. Dilation invariance of complex Brownian motion is more tricky as it requires rescaling time. Let $B_t = B_t^1 + iB_t^2$, $\lambda = x + iy$, $X_t = \lambda B_{t/|\lambda|^2}$. We see $X_0 = \lambda \cdot B_0$ is the starting point for the Brownian motion. Then we have

$$\lambda B_{t/|\lambda|^2} = (a+bi)(B^1_{t/|\lambda|^2} + iB^2_{t/|\lambda|^2}) = (a \cdot B^1_{t/|\lambda|^2} - b \cdot B^2_{t/|\lambda|^2}) + i(a \cdot B^2_{t/|\lambda|^2} + b \cdot B^1_{t/|\lambda|^2})$$

Looking at the real part of $X_t - X_s$ we get

$$Re(X_t - X_s) = (a \cdot B^1_{t/|\lambda|^2} - b \cdot B^2_{t/|\lambda|^2}) - (a \cdot B^1_{s/|\lambda|^2} - b \cdot B^2_{s/|\lambda|^2}) =$$

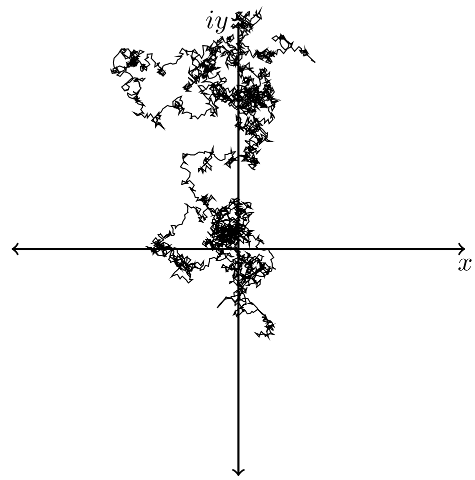
$$a \cdot (B^1_{t/|\lambda|^2} - B^1_{s/|\lambda|^2}) - b \cdot (B^2_{t/|\lambda|^2} - B^2_{s/|\lambda|^2}) \sim a \cdot N(0, \frac{t-s}{|\lambda|^2}) - b \cdot N(0, \frac{t-s}{|\lambda|^2})$$

$$\sim N(0, \frac{a^2(t-s)}{|\lambda|^2}) - N(0, \frac{b^2(t-s)}{|\lambda|^2}) \sim N(0, \frac{|\lambda|^2(t-s)}{|\lambda|^2}) = N(0, t-s)$$

as desired. A similar calculation follows for the imaginary part finishing the proof that X_t is a complex Brownian motion.

This result strongly suggests the full property of conformal invariance. Recall that a conformal map is a bijective holomorphic mapping whose derivative never vanishes. Although conformal maps may exhibit complicated global behavior, locally, they act as a combination of translation and dilation. However, proving the full result is quite tricky and requires developing the theory of stochastic calculus.

It is not necessarily obvious why such machinery is needed, so below is a graph depicting a complex Brownian motion which should help motivate why new techniques are required.



We see Brownian motion lacks many of the well-behaved properties associated with the smooth curves commonly used in complex analysis. It turns out that, while continuous, Brownian curves are unrectifiable and nowhere differentiable. In fact 1 dimensional Brownian motion serves as a nice example for an (almost surely) continuous nowhere differentiable function.

Proof. Let B_t be a 1 dimensional Brownian Motion. Let $M_t = \sup\{B_s : s < t\}$ denote the maximum value achieved by time t and $\tau_a = \inf\{t : B_t = a\}$ denote the first time $B_t = a$ (infinite if B_t is never a). We have that $\mathbb{P}(M_t > a) = 2\mathbb{P}(B_t > a)$.

$$\begin{aligned}
\mathbb{P}(M_t > a) &= \mathbb{P}(\tau_a < t) \\
&= \mathbb{P}(B_t - B_{\tau_a} > 0 \,|\, \tau_a < t) + \mathbb{P}(B_t - B_{\tau_a} < 0 \,|\, \tau_a < t) \\
&= 2\mathbb{P}(B_t - B_{\tau_a} > 0 \,|\, \tau_a < t) \\
&= 2\mathbb{P}(B_t - a > 0 \,|\, \tau_a < t) \\
&= 2\mathbb{P}(B_t > a \,|\, \tau_a < t) \\
&= 2\mathbb{P}(B_t > a)
\end{aligned} \tag{1}$$

Now assume for sake of contradiction that $dB/dt(x) = A$ for some time x. Then by the newton quotient we have that B_t is locally linear at x. Thus there exists ϵ', C such that for all $\epsilon < \epsilon'$: $|B_{x+\epsilon} - B_x| < C \cdot \epsilon$. Now we take the probability of this event and use the fact that increments of the same length have the same distribution to rewrite $\mathbb{P}(|B_{x+\epsilon} - B_x| < \epsilon\ C) = \mathbb{P}(|B_\epsilon - B_0| < \epsilon\ C)$. Notice however

$$\begin{aligned}
\mathbb{P}(M_\epsilon > \epsilon\ C) &= 2\mathbb{P}(B_\epsilon > \epsilon\ C) \\
&= 2\mathbb{P}(N(0, \epsilon) > \epsilon\ C) \\
&= 2\mathbb{P}(N(0, 1) > \sqrt{\epsilon}\ C)
\end{aligned} \tag{2}$$

Then by taking ϵ to zero, we get $\mathbb{P}(M_\epsilon > \epsilon \cdot C) = 2\mathbb{P}(N(0,1) > 0) = 1$. Thus we find that B_t is in fact almost surely not differentiable at x. $\square$

Ultimately, this is what makes working with these types of stochastic functions difficult - the traditional theory of calculus is not equipped to handle their instantaneous variations.

3 Stochastic Calculus

3.1 Total and Quadratic Variation

Definition 3.1. Total variation is an absolute measure of how much a function changes. The absolute variation of f at time t is:

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(\frac{k}{n}t)-f(\frac{k-1}{n}t)|$$

For a rectifiable curve $f:[0,t]\to\mathbb{C}$ this sum will be finite. However as Brownian motion is poorly behaved, this sum diverges. Consequently, we must turn to quadratic variation.

Definition 3.2. Quadratic variation is the square of how much a function changes in time. The quadratic variation of f at time t, denoted as $\langle f\rangle_t$ is:

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(\frac{k}{n}t)-f(\frac{k-1}{n}t)|^2$$

For continuously differentiable functions, quadratic variation is always zero (this comes from f' being bounded on $[0,t]$). However for Brownian motion we discover an important property.

Theorem 3.1. *For a Brownian motion B_t, we have*

$$\lim_{n\to\infty}\sum_{k=1}^{n}(B_{tk/n}-B_{t(k-1)/n})^2=t$$

or in other words, if we define $dB_t=\lim_{\epsilon\to 0}B_{t+\epsilon}-B_t$, then we get that $(dB_t)^2=dt$

Proof. We have $B_{tk/n}-B_{t(k-1)/n}\sim N(0,\frac{t}{n})$ thus

$$\lim_{n\to\infty}\sum_{k=1}^{n}(B_{tk/n}-B_{t(k-1)/n})^2\sim\lim_{n\to\infty}\sum_{k=1}^{n}N(0,\frac{t}{n})^2=\lim_{n\to\infty}n\cdot\frac{1}{n}\sum_{k=1}^{n}N(0,\frac{t}{n})^2$$

By the central limit theorem this is

$$\lim_{n\to\infty}n\cdot\mathbb{E}[N(0,\frac{t}{n})^2]=\lim_{n\to\infty}n\cdot\frac{t}{n}=t$$

□

This represents a fundamental difference between classical calculus and stochastic calculus. Say we Taylor expand

$$f(x+\epsilon)=f(x)+f'(x)\epsilon+\frac{f''(x)}{2}\epsilon^2+...$$

We get the formula $df=f(x+\epsilon)-f(x)=f'(x)\epsilon$ as $\epsilon\to 0$ because the higher orders of ϵ^n vanish much faster than ϵ so they can be ignored in the limit.

Trying the same approach for $f(B_t)$

$$f(B_{t+\epsilon})=f(B_t)+f'(B_t)(B_{t+\epsilon}-B_t)+\frac{f''(B_t)}{2}(B_{t+\epsilon}-B_t)^2+...$$

But now we have $(B_{t+\epsilon}-B_t)^2=dt$ as $\epsilon\to 0$ which does not vanish quickly enough to be ignored. This leads to $f(B_{t+\epsilon})-f(B_t)=f'(B_t)\cdot dB_t+\frac{1}{2}f''(B_t)dt$ as $\epsilon\to 0$ or in terms of differentials
$df=f'(B_t)\cdot dB_t+\frac{1}{2}f''(B_t)dt$

Further extending to a function $f(x, y)$, Taylor expansion yields

$$f(t+dt, B_t+dB_t) = f(t, B_t) + \frac{\partial f}{\partial x}dt + \frac{\partial f}{\partial y}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt^2 + \frac{\partial^2 f}{\partial x \partial y}dtdB_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}dB_t^2 + ...$$

But in the limit we have that dt^2 and $d_t \cdot dB_t \to 0$. Informally this is because we can think of $dB_t = O(dt^{1/2})$. Thus $dt \cdot dB_t = O(dt^{3/2})$ which vanishes at a higher rate than dt. The resulting formula is

$$f(t+dt, B_t+dB_t) - f(t, B_t) = \frac{\partial f}{\partial x}dt + \frac{\partial f}{\partial y}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}dt$$

as $\epsilon \to 0$. Rewriting and regrouping gives the final result

$$df = (\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2})dt + \frac{\partial f}{\partial y}dB_t$$

For functions of complex Brownian motion $f(x, iy) \to \mathbb{R}$ we have two Brownian inputs. Letting B_t^1 and B_t^2 be 1 dimensional Brownian motion we get

$$f(B_t^1 + dB_t^1, B_t^2 + dB_t^2) =$$

$$f(B_t^1, B_t^2) + \frac{\partial f}{\partial x}dB_t^1 + \frac{\partial f}{\partial y}dB_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dB_t^1)^2 + \frac{\partial^2 f}{\partial x \partial y}dB_t^1 dB_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dB_t^2)^2 + \dots$$

Again taking ϵ to zero and using $(dB_t)^2 = dt$ we get

$$df = \frac{\partial f}{\partial x}dB_t^1 + \frac{\partial f}{\partial y}dB_t^2 + \frac{1}{2}(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})dt + \frac{\partial^2 f}{\partial x \partial y}dB_t^1 dB_t^2$$

With this theory developed, we are ready to begin the proof of conformal invariance.

4 Lévy's theorem on conformal invariance

Before we can fully complete the proof we need one more tool, the Doubins-Swartz theorem.

Theorem 4.1. *If X_t is a local martingale and $\lim_{t\to\infty}\langle X\rangle_t = \infty$ then we can define $\sigma_s = \inf\{s : \langle X\rangle_t > s\}$ and $B_t = X_{\sigma_t}$ is a Brownian motion with $X_t = B_{\langle X\rangle_t}$*

The proof is a slightly intricate and involves applying the optional stopping theorem to the expression $exp(iy\, X_t + y^2\, \langle X\rangle_t/2)$ in order to show increments are normally distributed. For a comprehensive explanation of this proof, please refer to Section 1.7 in Lawler's book. Having covered that final preliminary, we can now begin the proof of Lévy's theorem.

Theorem 4.2. *Let $f : U \to V$ be a conformal map and B_t be a complex Brownian motion starting at z_0 restricted to U. Then there exists $\tilde{B}_t$ in V started at $f(z_0)$ for which $f(B_t) = \tilde{B}_{\int_0^t |f'(B_s)|^2\, ds}$*

Proof. Let $U, V \subset \mathbb{C}$, $f : U \to V$ be a conformal map, and $B_t^{z_0} = B_t^1 + iB_2^t$ be a complex Brownian motion starting at z_0. We can rewrite $f(x+iy)$ as $u(x, y) + iv(x, y)$. From our previous work we have that

$$du(B_t^1, B_t^2) = \frac{\partial u}{\partial x}dB_t^1 + \frac{\partial u}{\partial y}dB_t^2 + \frac{1}{2}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})dt + \frac{\partial^2 u}{\partial x \partial y}dB_t^1 dB_t^2$$

But as u is harmonic as it is the real part of a holomorphic function, thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Furthermore, we also have that $dB_t^1\, dB_t^2 = 0$.

$$dB_t^1 \cdot dB_t^2 = \lim_{\epsilon\to 0}(B_{t+\epsilon}^1 - B_t^1)(B_{t+\epsilon}^2 - B_t^2)$$

The expectation of this is

$$\mathbb{E}[(B^1_{t+\epsilon} - B^1_t)] \cdot \mathbb{E}[(B^2_{t+\epsilon} - B^2_t)] = \mathbb{E}[N(0, \epsilon)] \cdot \mathbb{E}[N(0, \epsilon)] = 0$$

Similarly the variance is

$$Var(B^1_{t+\epsilon} - B^1_t)(B^2_{t+\epsilon} - B^2_t) =$$
$$\mathbb{E}[(B^1_{t+\epsilon} - B^1_t)^2]\, \mathbb{E}[(B^2_{t+\epsilon} - B^2_t)^2] - \mathbb{E}[(B^1_{t+\epsilon} - B^1_t)]^2\, \mathbb{E}[(B^2_{t+\epsilon} - B^2_t)]^2 =$$
$$\mathbb{E}[(N(0, \epsilon)^2]\, \mathbb{E}[(N(0, \epsilon)^2] = \epsilon^2$$

Thus as $\epsilon \to 0$ we get $dB^1_t\, dB^2_t = 0$ almost surely as it has zero variance and zero expectation. Returning to du we ultimately get

$$du(B^1_t, B^2_t) = \frac{\partial u}{\partial x} dB^1_t + \frac{\partial u}{\partial y} dB^2_t$$

and by similar calculation

$$dv(B^1_t, B^2_t) = \frac{\partial v}{\partial x} dB^1_t + \frac{\partial v}{\partial y} dB^2_t$$

The remainder of this proof relies on the Doubins-Swartz theorem. We first get that u is a martingale as it has no dt component. More generally if $df = \mu(B_t)dt + \sigma(B_t)dB_t$ then f a martingale if and only if $\mu = 0$ identically. This informally is because μ acts as a drift term which biases f. Now we must observe the quadratic variation of u.

$$\langle u \rangle_t = \lim_{n\to\infty} \sum_{k=1}^{n} (du_{t_k})^2$$

Here $\epsilon = 1/n$ and $t_k = k\, \epsilon \cdot t$.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{\partial u}{\partial x}(B_{t_k}) dB^1_t)^2 + 2(\frac{\partial u}{\partial x}(B_{t_k}) \cdot \frac{\partial^u}{\partial y}(B_{t_k})) dB^1_t\, dB^2_t + (\frac{\partial u}{\partial y}(B^2_{t_k}) dB^2_t)^2$$

And as we established $dB^1_t\, dB^2_t = 0$ we get

$$\langle u \rangle_t = \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{\partial u}{\partial x}(B_{t_k}) dB^1_t)^2 + (\frac{\partial u}{\partial y}(B_{t_k}) dB^2_t)^2 = \int_0^t [\frac{\partial u}{\partial x}(B_s)]^2 + [\frac{\partial u}{\partial y}(B_s)]^2\, ds =$$

$$= \int_0^t [\frac{\partial u}{\partial x}(B_s)]^2 + [-\frac{\partial v}{\partial x}(B_s)]^2\, ds = \int_0^t |f'(B_s)|^2 ds$$

For convenience we just adopt the standard notation of an integral as the limit of a sum. It is also clear to see that the same calculation shows a similar result for $\langle v \rangle_t$. As f is conformal its derivative never vanishes so we get that $\langle u \rangle_t \to \infty$ as $t \to \infty$. For example if U is compact we get that f' has a minimum magnitude M and thus $\langle u \rangle_t \geq t \cdot M^2$ is unbounded.

We can finally apply Doubins-Swartz. Thus letting $\sigma(t) = \int_0^t |f'(B_s)|^2 ds$, we get $u(B_t) = \tilde{B}^1_{\sigma(t)}$ is a Brownian motion. Similarly $v(B_t) = \tilde{B}^2_{\sigma(t)}$ is also Brownian motion. Together this shows $f(B_t) = u(B_t) + i\, v(B_t) = \tilde{B}^1_{\sigma(t)} + i\, \tilde{B}^2_{\sigma(t)}$ is complex Brownian motion. It is also clear that $\tilde{B}_{\sigma(0)} = f(B_0) = f(z_0)$ completing the proof.

An aside about Doubins-Swartz: without going into technicality, local martingales appear if B_t leaves U. In this case, $f(B_t)$ is not defined. Consequently, we have $f(B_t)$ is only Brownian motion for the time when B_t is inside U. We call both the stopped versions of B_t and $f(B_t)$ local martingales. $\square$

This proof truly highlights how rare functional invariance of Brownian motion is. In fact it seems limited almost exclusively to holomorphic functions. Why? In the proof we required both u and v to be harmonic in order to cancel the dt term when deriving du and dv. Even further, we see that not just any pair of harmonic functions will work (much in the same way not just any pair of harmonic functions make a holomorphic function). We also require that the quadratic variation of u and v match in order to get the time scaling for the real and imaginary dimensions to match. This restricts u and v to either be equal or harmonic conjugates. We require f to be conformal in order to ensure that the quadratic variation is almost surely strictly increasing but the theorem also holds so long f is a non constant holomorphic function. This is because the zeros of f' can never accumulate which makes them largely ignorable.

5 Liouville's theorem

The framework of complex Brownian motion allows us to tackle problems in a new probabilistic way. For example we can prove Liouville's theorem using only properties of complex Brownian motion. Recall, Liouville's theorem states that any bounded entire function is constant.

Proof. Assume for sake of contradiction that f is a bounded, nonconstant, entire function. We get for some Brownian motion B_t that $f(B_t)$ is also a Brownian motion for all time t. And as f is bounded we get that the Brownian motion $f(B_t)$ is also bounded for all t. But this is a contradiction as Brownian motion is almost surely unbounded. Let M be an arbitrary bound. Clearly $B_t^1 \sim N(0,t)$. Thus

$$\begin{aligned}\mathbb{P}(-M < B_t^1 < M) &= \Phi\left(\frac{M}{\sqrt{t}}\right) - \Phi\left(\frac{-M}{\sqrt{t}}\right) \\ &= \Phi\left(\frac{M}{\sqrt{t}}\right) - \left(1 - \Phi\left(\frac{M}{\sqrt{t}}\right)\right) \\ &= 2\Phi\left(\frac{M}{\sqrt{t}}\right) - 1\end{aligned}$$

Here Φ is the CDF of $N(0,t)$ and we take advantage of $\Phi(-x) = 1 - \Phi(x)$. Then by letting $t \to \infty$ we get $\mathbb{P}(-M < B_t^1 < M) = 2\Phi(0) - 1 = 0$ concluding that B_t^1 is almost surely unbounded.

Thus we conclude that f is constant. $\square$

6 Conclusion

Lévy's theorem on conformal invariance is a remarkable example of complex analysis appearing in seemingly unrelated branches of mathematics. This theorem represents a bridge between complex analysis and the theory of stochastic processes. It not only provides a deeper understanding of the nature of Brownian motion but also offers a new tool to tackle purely complex analytic questions. The connections revealed by Lévy's theorem highlight the rich interplay between different areas of mathematics.

7 Acknowledgments

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References

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