# Hartogs Extension Theorem

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## 1 Introduction

This paper covers Hartogs Extension Theorem. This result was seminal discovery made in 1906 by German mathematician Friedrich Hartogs. The statement proves holomorphic functions of several variables can be extended inwards in a qualitatively different way than holomorphic functions of one variable. This discovery stands as one of the earliest demonstrations of the substantial inherent differences between complex analysis in several variables and its single-variable counterpart.

## 2 Holomorphicity in Higher Dimensions

Before exploring the theorem, it's necessary to expand our current theory to multiple variables. The definitions and proof of the extension principle up to section 3 follow information outlined in [1, Chapter 1]. Let's first look at the single variable case. If $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$. We say that $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic iff it is continuous and satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In several variables we say a function is separately holomorphic if the Cauchy Riemann equations hold for each variable. This is to say Let $U \subset \mathbb{C}^n$ open, and $f : U \to \mathbb{C}$ then $f(z_1, ..., z_n)$ is holomorphic iff for all $i$, $z_i = x_i + y_i$ we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}$$

And we then say $f$ is holomorphic if it is separately holomorphic and continuous. This turns out to be a natural way to extend the Cauchy-Riemann equations as it preserves many theorems true in one dimension. One notable example of this is The Cauchy Integral Formula for Polydiscs.

### 2.1 Cauchy Integral Formula for Polydiscs

A natural basis for the topology of $\mathbb{C}^n$ is the polydisc. A polydisc centered at $w \in \mathbb{C}^n$ with radius $\epsilon \in \mathbb{R}^n$ is defined as

$$B_\epsilon(w) = \{z \in C^n : |z_i - w_i| < \epsilon_i\}$$

Then to extend Cauchy's integral formula we have

**Theorem 1** (Cauchy Integral Formula for Polydiscs)**.** *Let $f\colon \overline{B_\epsilon(w)} \to \mathbb{C}$ be a holomorphic function. Then for any $z \in B_\epsilon(w)$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - z_1| = \epsilon_1} \cdots \int_{|\zeta_n - z_n| = \epsilon_n} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}\, d\zeta_n \cdots d\zeta_1$$

This follows directly from treating the integrand as a single variable holomorphic function of $\zeta_n$ and repeatedly applying the one dimensional Cauchy Integral Formula. As expected, there is also a high order Integral Formula which follows from the single variable case. Let $\alpha = \alpha_1 + \cdots + \alpha_n$, then we have that

$$\frac{\partial^\alpha f}{\partial^{\alpha_1} z_1 \ldots \partial^{\alpha_n} z_n}(w) = \frac{\alpha_1! \cdots \alpha_n!}{(2\pi i)^n} \int_{|\zeta_1 - z_1| = \epsilon_1} \cdots \int_{|\zeta_n - z_n| = \epsilon_n} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - w_1)^{\alpha_1 + 1} \cdots (\zeta_n - w_n)^{\alpha_n + 1}}\, d\zeta_n \cdots d\zeta_1$$

### 2.2 Holomorphic Functions are Analytic

Exactly as with one variable, we can use the Cauchy Integral Formula to show that holomorphic functions are analytic. Suppose: $U \subset \mathbb{C}^n$ open and $f : U \to \mathbb{C}$ holomorphic. Then

$$\forall w \in U, \exists B_\epsilon(w) : f(z) = \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1, \ldots, k_n} (z_1 - w_1)^{k_1} \cdot (z_2 - w_2)^{k_2} \cdots (z_n - w_n)^{k_n}$$

For all $z \in B_\epsilon(w)$ and

$$a_{k_1, \ldots, k_n} = \frac{1}{k_1! \cdots k_n!} \cdot \frac{\partial^k f}{\partial^{k_1} z_1 \ldots \partial^{k_n} z_n}(w)$$

This allows for many similar results as from the lower dimensional case. The proofs for the Maximum Principle and Identity Theorem (for open subsets) transfer without much change.

## 3 Hartogs Extension Theorem Proof

Before exploring the statement and proof of Hartogs Theorem in depth, we must establish one last lemma.

**Lemma 1.** *Let $U \subset \mathbb{C}^n$ be open and $V \subset \mathbb{C}$ be an open neighbourhood of the boundary $\partial B_\epsilon(0) \subset \mathbb{C}$. If $f : V \times U \to \mathbb{C}$ is holomorphic then*

$$g(z) \coloneqq g(z_1, \ldots, z_n) \coloneqq \int_{|\zeta|=\epsilon} f(\zeta, z_1, \ldots, z_n) d\zeta$$

*is a holomorphic function on $U$.*

This result allows us to classify functions expressed as an integral of a known holomorphic function as holomorphic which will be a critical step in the proof of Hartogs Theorem.

*Proof.* Let $z \in U$, $\zeta \in \partial B_\epsilon(0) \subset \mathbb{C}$. Clearly $(z, \zeta) \in V \times U$. We can then take advantage of $f$ being analytic to define a power series on some polydisc $B_{\delta(\zeta)}(\zeta) \times B_{\delta'(\zeta)}(z) \subset V \times U$.

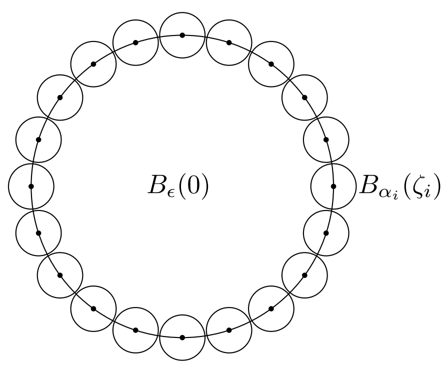
We also have that $\partial B_\epsilon(0)$ is compact and we can form an open cover with

$$\bigcup_{\zeta \in \partial B_\epsilon(0)} B_\epsilon(0) \cap B_{\delta(\zeta)}(\zeta)$$

Therefore we can generate a finite subcover $\{B_\epsilon(0) \cap B_{\delta(\zeta_i)}(\zeta_i)\}_{i=1}^n$ with $\zeta_i \in \partial B_\epsilon(0)$ and $\delta(\zeta_i) \in \mathbb{R}$. Then we can select $\alpha_i < \delta(\zeta_i)$ to form a finite "near" subcover. That is to say $\{B_\epsilon(0) \cap B_{\alpha_i}(\zeta_i)\}_{i=1}^n$ is pairwise disjoint and

$$\partial B_\epsilon(0) = \bigcup_{i=1}^n (\partial B_\epsilon(0) \cap \overline{B_{\alpha_i}(\zeta_i)})$$

This is visualized below




As there are only finitely many points in $B_\epsilon(0)$ that are not in the union of discs, we have that

$$g(z) = \int_{|\zeta|=\epsilon} f(\zeta, z_1, \ldots, z_n) = \sum_{i=1}^n \int_{|\zeta|=\epsilon,\ |\zeta_i - \zeta| < \alpha_i} f(\zeta, z) d\zeta$$

Recall however that $f$ is analytic. Its power series converges uniformly and absolutely on $\overline{B_{\alpha_i}(\zeta_i)}$ for fixed $w \in U$. Thus $\forall w \in U$, we can locally expand $f$ around $(\zeta_i, w)$. Putting this all together yields:

$$g(z_1, \ldots, z_n) = \sum_{i=1}^{n} \int_{|\zeta|=\epsilon,\ |\zeta_i - \zeta| < \alpha_i} f(\zeta, z) d\zeta =$$

$$\sum_{i=1}^{n} \int_{|\zeta|=\epsilon,\ |\zeta_i - \zeta| < \alpha_i} \sum_{k_0, \ldots, k_n = 0}^{\infty} a_{k_0, \ldots, k_n} (\zeta - \zeta_i)^{k_0} \cdot (z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n} d\zeta =$$

$$\sum_{i=1}^{n} \sum_{k_0, \ldots, k_n = 0}^{\infty} \left[ \int_{|\zeta|=\epsilon,\ |\zeta_i - \zeta| < \alpha_i} a_{k_0, \ldots, k_n} (\zeta - \zeta_i)^{k_0} d\zeta \right] \cdot (z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n} =$$

$$\sum_{k_0, \ldots, k_n = 0}^{\infty} \left[ \sum_{i=1}^{n} \int_{|\zeta|=\epsilon,\ |\zeta_i - \zeta| < \alpha_i} a_{k_0, \ldots, k_n} (\zeta - \zeta_i)^{k_0} d\zeta \right] \cdot (z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}$$

We can clearly see $g$ is holomorphic through its power series. $\square$

With the preliminary established we can now move on to the statement and proof of Hartogs Theorem

**Theorem 2** (Hartogs Extension Theorem). *Suppose $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_n)$ such that for all $i : \epsilon'_i < \epsilon_i$. Then if $n > 1$ then any holomorphic function $f : B_\epsilon(0) \setminus \overline{B_{\epsilon'}(0)} \to \mathbb{C}$ can be uniquely extended to a holomorphic function $f : B_\epsilon(0) \to \mathbb{C}$.*

*Proof.* There is $\delta > 0$ such that $V := \{z : \epsilon_1 - \delta < |z_1| < \epsilon_1,\ |z_{i \neq 1}| < \epsilon_i\} \cup \{z : \epsilon_2 - \delta < |z_2| < \epsilon_2, |z_{i \neq 2}| < \epsilon_i\} \subset B_\epsilon(0) \setminus \overline{B_{\epsilon'}(0)}$. Thus $f$ is holomorphic on $V$. If we fix $w = (w_2, \ldots, w_n)$, we can think of $f_w(z_1) = f(z_1, w_2, \ldots, w_n)$ as a holomorphic function function of $z_1$ on $\{z : \epsilon_1 - \delta < |z| < \epsilon_1\}$. Thus we can write $f_w(z_1)$ using it's Laurent Series.

$$f_w(z_1) = \sum_{-\infty}^{\infty} a_n(w) \cdot z_1^n$$

with

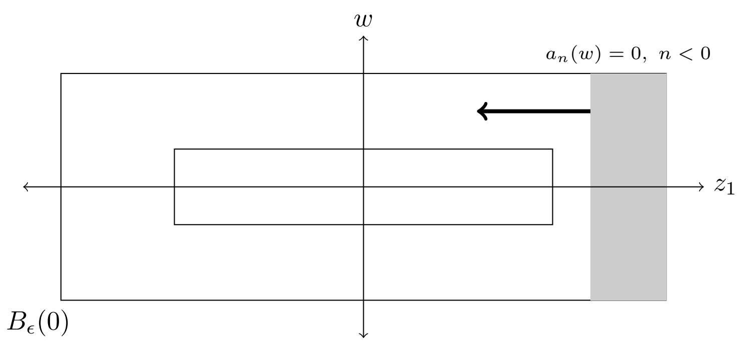
$$a_n(w) = \frac{1}{2\pi i} \int_{|\zeta| = \epsilon_1 - \delta/2} \frac{f_w(\zeta)}{\zeta^{n+1}} d\zeta$$

and by Lemma 1, we have that $a_n(w)$ is holomorphic on $B_{(\epsilon_2, \ldots, \epsilon_n)}(0)$. We also have that $f_w(z_1)$ is holomorphic on $\{\ z : |z| < \epsilon_1\}$ when we fix $w$ such that $\epsilon_2 - \delta < |w_2| < \epsilon_2$. Thus the Laurent series of $f$ on the region is really a power series, and $a_n(w) = 0$ for $n < 0$. But by the Identity Theorem, we have that $a_n(w) = 0$ for all $n < 0$.This allows us to define our holomorphic extension

$$\hat{f}(z, w) = \sum_{n=0}^{\infty} a_n(w) \cdot z_1^n$$

We have that $a_n(w)$ is holomorphic thus achieves a maximum on the boundary. Furthermore, the series converges uniformly on the edges, so it most also converge uniformly as we extend the function inwards. It's also clear that our extension agrees with $f$ on $B_\epsilon(0) \setminus \overline{B'_\epsilon(0)}$. And this extension is trivially unique due to the Identity Theorem. $\square$

Here is a rough diagram to aid the proof




The proof first establishes that we can write $f_w(z_1)$ as a Laurent Series. But as we can see in the picture, near the edge of $B_\epsilon(0)$, we have $f_w(z_1)$ is defined on a disc, and thus the negative terms of the Laurent series vanish there. However, we know that the Laurent coefficients are holomorphic in $w$, and so by the identity theorem, the negative terms must vanish everywhere. Then we can use resulting the power series to holomorphically extend inwards.

## 3.1 Contrasting with single variable functions

It is important to note that this result only holds when $n$ is strictly greater than 1. We can see that this is not at all the case in one dimension with a function like $f(z) = 1/z$. It can clearly be holomorphic on an annulus around 0, but there is no way to holomorphically extend it.

If we try extending this counterexample into two dimensions: $f(z_1, z_2) = 1/z_1$ we see that instead of a point as before, there is now a singular curve $(0, z_2)$ which will intersect any annulus containing it. Trying a bit harder, we can see $f(x, y) = 1/(x^2 + y^2)$ has an isolated singularity at $(0, 0)$ for real $x, y$. Translating this to $f(z_1, z_2) = 1/(z_1^2 + z_2^2)$ fails however because $z^2$ is no longer positive definite. Plugging in $z_2 = i \cdot z_1$ still yields a curve of singularities.

This reveals a rather significant qualitative difference between holomorphic functions of one variable and several. Both the (non removable) singularities and zeros for high dimensional holomorphic functions must be unbounded. If there was a function with bounded singularities, it could be translated to the origin, and then would directly be a counterexample Hartogs Theorem. If there was a holomorphic function with bounded zeros, then the previous process could be applied to it's inverse. As we can see Hartogs Theorem turns out to be quite strong already, but it can actually be further strengthened.

# 4 Stronger Claims and Implications

One obvious weakness of the Extension Principle as stated is that it is only defined on annular polydiscs. This can be broadened to the more general statement.

**Theorem 3.** *Let $\Omega \subset \mathbb{C}^n, (n > 1)$ be open, connected, and bounded. Let $K \subset \Omega$ be a compact subset, such that $\Omega \setminus K$ is still open and connected. Then every holomorphic function $f : \Omega \setminus K \to \mathbb{C}$ has a unique holomorphic extension $\hat{f} : \Omega \to \mathbb{C}$ such that $\hat{f}|_{\Omega \setminus K} = f$*

This conveniently generalizes $B_\epsilon(0)$ to any bounded domain within $\mathbb{C}^n$ and the undefined region within to be any compact subset still preserving connectedness. The proof of this statement turns out to be significantly more involved than the case involving polydiscs. You can find two different techniques for approaching the proof in [2] and [3, Theorem 7]. Very roughly speaking though, [2] proves this by working locally and gluing many small extension together. [3] approaches the statement using partial differential equations in order to sidestep the topological challenges addressed by [2]. Both are worth looking at. This stronger definition is also equivalent to the following

**Theorem 4** (Bochner's Extension Theorem)**.** *Let $\Omega \subset \mathbb{C}^n, (n > 1)$ be open, connected, and bounded and $\mathbb{C}^n \setminus \Omega$ connected. Then every holomorphic function defined on an open neighborhood of the boundary, $U \supset \partial\Omega, f : U \to \mathbb{C}$, has a unique holomorphic extension $\hat{f} : \Omega \to \mathbb{C}$ such that $\hat{f}|_U = f$*

*Proof.* To show Theorem 3 implies Theorem 4, let $\Omega$ be open, connected and bounded and $U \supset \partial\Omega$ be open. $\Omega \setminus U$ must be closed and bounded, thus compact. We can also see that $\Omega \setminus K = \Omega \cap U$ is open, connected and bounded. Thus we can apply Theorem 3, getting a holomorphic extension to any $f$.

To show Theorem 4 implies Theorem 3, let $\Omega$ be open, connected and bounded and $K \subset \Omega$ be compact, such that $\Omega \setminus K$ is still open and connected. As $\Omega$ open, there is an open neighborhood, $U : K \subset U \subset \Omega$. $U$ is open thus doesn't contain it's boundary points, yet $U$ contains all points of $K$, so we clearly have $\partial U \subset \Omega \setminus K$. Finally we can apply Theorem 4 to $\Omega \setminus K$ as $\Omega$ is open. connected and bounded, $\mathbb{C}^n \setminus \Omega$ is connected. Therefore any holomorphic function defined on $\Omega \setminus K$ can be holomorphically extended to $\Omega$. $\square$

Again this is Theorem is unique to holomorphic functions of several variables. There is generally no expectation of being able to extend holomorphic functions of one variable, and the theorems that do allow extensions have much stronger requirements. Riemann's Removable Singularity Theorem for instance requires $f$ to be locally bounded in order to extend to just a single point.

Bochner's Extension Theorem highlights a general theme of holomorphic functions, namely that their behavior near the boundary dictates their global behavior on the interior. The Maximum Principle tells us that the maximum modulus of a holomorphic function must occur on the boundary. Stronger yet, Cauchy's Integral Formula tells us that knowing the values of a holomorphic function on the boundary of a (poly)disc entirely dictate the values in the interior. In a similar vein, Bochner's Extension Principle states that merely knowing a holomorphic function is defined on an arbitrarily small neighborhood of the boundary is enough to know it can be holomorphically extended into the interior.

Bochner's Theorem still requires a function to be defined on a full neighborhood of $\partial\Omega$. What if instead we only have a function defined on $\partial\Omega$? The main issue here is that the current definition of being holomorphic fails. Being defined on a neighborhood of the boundary allows for all partial derivatives to be taken, whereas only being defined on the boundary restricts the direction you can take limits. To get around this we instead require that $f$ satisfy the tangential Cauchy-Riemann equations. You can think of this as requiring $f$ to be holomorphic within the tangential directions it is "allowed to move". Such a function is called a CR function. This leads us to our final theorem.

**Theorem 5.** *Let $\Omega \subset \mathbb{C}^n, (n > 1)$ be open, connected, and bounded. Let $\partial\Omega$ be $C^1$ and connected. If $f : \partial\Omega \to \mathbb{C}$ is a CR function, then there is an extension $\hat{f} : \overline{\Omega} \to \mathbb{C}$ such that $\hat{f}$ is continuous on $\overline{\Omega}$, holomorphic on $\Omega$, and $\hat{f} = f$ on $\partial\Omega$.*

The rigorous background of CR functions and proof of this statement are beyond the scope of this paper but more detailed explanations can be found in [3, Theorem 9]. Regardless of not having the tools to prove it, this result feels like a rather natural continuation of Bochners Extension Theorem as we could already make $U \supset \partial\Omega$ arbitrarily thin. Taking the next step to Theorem 5 and only requiring $f$ be defined on $\partial\Omega$ further emphasizes an important principle in complex analysis: the way a holomorphic function acts on the boundary determines its behavior in the interior.

# 5 Conclusion

In this paper's exploration of Hartogs Extension Theorem, we've proved the basic result and highlighted its fundamental importance in the theory of several complex variables. Our discussion, spanning from foundational concepts to advanced generalizations, illustrates the theorem's pivotal role in understanding the unique properties of multi-variable holomorphic functions. Hartogs theorem and its logical extensions underscore the significant relationship between the boundary behavior of holomorphic functions and their interior behavior - a recurring theme within complex analysis.

## 6 Acknowledgments

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## References

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